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The strong law of large numbers for random quadratic forms

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Published in:
Theory of probability and its applications

DOI:
[10.1137/1140005](https://doi.org/10.1137/1140005)

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
1996

[Link to publication in University of Groningen/UMCG research database](#)

Citation for published version (APA):

Mikosch, T. (1996). The strong law of large numbers for random quadratic forms. *Theory of probability and its applications*, 40(1), 76-91. <https://doi.org/10.1137/1140005>

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ON THE STRONG LAW OF LARGE NUMBERS FOR RANDOM QUADRATIC FORMS*

T. MIKOSCH†

Abstract. The paper establishes strong laws of large numbers for the quadratic forms

$$Q_n(X, X) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} X_i X_j$$

and the bilinear forms

$$Q_n(X, Y) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} X_i Y_j,$$

where $X = (X_n)$ is a sequence of independent random variables and $Y = (Y_n)$ is an independent copy of it. In the case of independent identically distributed symmetric p -stable random variables X_n we derive necessary and sufficient conditions for the strong laws of $Q_n(X, X)$ and $Q_n(X, Y)$ for a given nondecreasing sequence (b_n) of normalizing constants. For these classes of variables (X_n) the strong laws $\lim b_n^{-1} Q_n(X, X) = 0$ a. s. and $\lim b_n^{-1} Q_n(X, Y) = 0$ a. s. are shown to be equivalent provided that $a_{ii} = 0$ for all i .

Key words. quadratic forms, bilinear forms, strong law of large numbers, Prokhorov-type characterization, p -stable random variables, domains of partial attraction, tail probabilities

Introduction. In this paper we study the strong law of large numbers (SLLN) for the quadratic forms (q. f.'s) $Q_n(X, X) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} X_i X_j$ and the bilinear forms (b. f.'s) $Q_n(X, Y) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} X_i Y_j$, where $X = (X_n)$ is a sequence of independent random variables (r. v.'s), $Y = (Y_n)$ is an independent copy of X , and (a_{ij}) is a symmetric (i. e., $a_{ij} = a_{ji}$) double array of real numbers.

The almost sure (a. s.) convergence of q. f.'s and b. f.'s has been intensively studied during the past few years. We refer to works of Varberg [28], [29], Sjögren [25], Cambanis et al. [3], Krakowiak and Szulga [10], and Kwapień and Woyczynski [11]. These authors derived necessary and sufficient conditions for the a. s. convergence of $Q_n(X, X)$ and $Q_n(X, Y)$ expressed by certain characteristics of the distribution of the X_n and by the coefficients a_{ij} . They also showed that $Q_n(X, X)$ converges a. s. if and only if both $Q_{n1}(X, X) = \sum_{i=1}^n a_{ii} X_i^2$ and $Q_{n2}(X, X) = 2 \sum_{i=2}^n X_i \sum_{j=1}^{i-1} a_{ij} X_j$ converge a. s.

The SLLN was studied by Wilmesmeyer and Wright [30], [31] and Szulga and Woyczynski [27]. Using martingale methods they proved SLLN's for both $Q_{n1}(X, X)$ and $Q_{n2}(X, X)$ and combined them to get an SLLN for $Q_n(X, X)$. For work concerning the law of the iterated logarithm for $Q_n(X, X)$ we refer to Fernholz and Teicher [6] and Mikosch [16]–[20]. A fairly general 0–1 law for q. f.'s in Gaussian r. v.'s was proved by de Acosta [1].

In this paper we want to derive necessary and sufficient conditions for the SLLN's $\lim b_n^{-1} Q_n(X, X) = 0$ a. s. and $\lim b_n^{-1} Q_n(X, Y) = 0$ a. s. for a given sequence (b_n) of positive numbers satisfying $b_n \uparrow \infty$.

In § 1 we introduce notation and basic facts applied throughout. Among others, we recall the definitions of p -stable random variables and what it means that an r. v.

*Received by the editors May 8, 1991.

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Z does not belong to the domain of partial attraction of the normal law (in short $Z \in \text{DPA}(\mathcal{N})$). We also give some useful estimates of tail probabilities for q.f.'s in Gaussian r.v.'s and recall some algebra of matrices and their eigenvalues.

In §2 we deal with independent symmetric r.v.'s X_n . We show that

$$\lim b_n^{-1} Q_n(X, X) = 0 \quad \text{a.s.}$$

holds if and only if

$$\lim b_n^{-1} Q_{n1}(X, X) = 0 \quad \text{a.s.}$$

and

$$\lim b_n^{-1} Q_{n2}(X, X) = 0 \quad \text{a.s.}$$

In §3 we consider independent identically distributed (i.i.d.) r.v.'s X_n . For symmetric $X_n \in \text{DPA}(\mathcal{N})$ and for certain p -stable nonsymmetric X_n with $p < 2$ we prove that the SLLN $\lim b_n^{-1} Q_n(X, X) = 0$ a.s., the a.s. convergence of the series $\sum_{i=1}^{\infty} b_i^{-1}(Q_i - Q_{i-1})$ and the relation $\lim b_n^{-1}(Q_n - Q_{n-1}) = 0$ a.s. are equivalent. Moreover, for these classes of X_n the SLLN $\lim b_n^{-1} Q_n(X, X) = 0$ a.s. holds if and only if $\lim b_n^{-1} Q_{n2}(X, Y) = 0$ a.s. The convergence of $\sum_{i=1}^{\infty} b_i^{-1}(Q_i - Q_{i-1})$ can be described by certain nonrandom characteristics introduced in [3], [10], and [11].

It should be noted that the a.s. behavior of q.f.'s in i.i.d. symmetric r.v.'s X_n with $X_n \notin \text{DPA}(\mathcal{N})$ is very much like the behavior of weighted sums. Indeed, in this case the SLLN $\lim b_n^{-1} \sum_{i=1}^n w_i X_i = 0$ a.s. for arbitrary weights (w_i) the a.s. convergence of the series $\sum_{i=1}^{\infty} b_i^{-1} w_i X_i$ and the relation $\lim b_n^{-1} w_n X_n = 0$ a.s. are equivalent (cf., e.g., [21], [22]).

In §4 we study the SLLN for q.f.'s and b.f.'s in i.i.d. $\mathcal{N}(0, 1)$ r.v.'s. Our main result shows that the SLLN $\lim b_n^{-1} Q_n(X, X) = 0$ a.s. holds if and only if

$$(0.1) \quad \lim 2^{-k} \sum_{i,j=n_{k-1}+1}^{n_k} a_{ij} X_i X_j = 0 \quad \text{a.s.},$$

$$(0.2) \quad \lim 2^{-k} \sum_{i=n_{k-1}+1}^{n_k} \sum_{j=1}^{n_{k-1}} a_{ij} X_i X_j = 0 \quad \text{a.s.}$$

and $\lim b_n^{-1} \sum_{i=1}^n a_{ii} = 0$. Here $n_k = \max\{n: b_n \leq 2^k\}$. In this way, we get a Prokhorov-type characterization of the SLLN. For independent symmetric r.v.'s Z_n Prokhorov [24] proved that the SLLN $\lim n^{-1} \sum_{i=1}^n Z_i = 0$ a.s. holds if and only if $\lim 2^{-k}(Z_{2^{k-1}+1} + \cdots + Z_{2^k}) = 0$ a.s. The block sums $Z_{2^{k-1}+1} + \cdots + Z_{2^k}$ are independent and easier to handle than the whole sum $\sum_{i=1}^n Z_i$. The same is true for the "block" q.f.'s in (0.1) and (0.2). Using Borel–Cantelli arguments and exponential tail estimates for Q_n we get conditions implying (0.1) and (0.2). An essential tool for the proofs of this section is the fact that the SLLN's $\lim b_n^{-1}(Q_n(X, X) - \mathbf{E}Q_n(X, X)) = 0$ a.s. and $\lim b_n^{-1} Q_n(X, Y) = 0$ a.s. are equivalent.

In §5 we give the proofs of the results formulated in §4.

1. Notation, definitions, basic facts. By $X = (X_n)$ we denote a sequence of real independent r.v.'s defined on a probability space $[\Omega, \mathcal{F}, \mathbf{P}]$; $Y = (Y_n)$ is an independent copy of X . Let (b_n) be a given sequence with $b_n \uparrow \infty$ and (a_{ij}) a symmetric double array of real numbers, i.e., $a_{ij} = a_{ji}$ for all i, j .

Notation. In what follows we put $\sum_{i=a}^b a_i = 0$ provided that $a > b$. For real sequences $x = (x_n)$ and $y = (y_n)$ we introduce

$$\begin{aligned} Q_{n1}(x, y) &= \sum_{i=1}^n a_{ii} x_i y_i, & Q_{n2}(x, y) &= \sum_{i=2}^n (x_i V_i(y) + y_i V_i(x)), \\ V_i(x) &= \sum_{j=1}^{i-1} a_{ij} x_j, & \overline{Q}_{n1}(x, y) &= \sum_{i=1}^n b_i^{-1} a_{ii} x_i y_i, \\ \overline{Q}_{n2}(x, y) &= \sum_{i=2}^n b_i^{-1} (x_i V_i(y) + y_i V_i(x)), \\ Q_n(x, y) &= Q_{n1}(x, y) + Q_{n2}(x, y), & \overline{Q}_n(x, y) &= \overline{Q}_{n1}(x, y) + \overline{Q}_{n2}(x, y). \end{aligned}$$

In case $y = x = X$ we suppress the dependence on X in the notation, i.e., we write $Q_n, Q_{n1}, Q_{n2}, \dots$ instead of $Q_n(x, x), Q_{n1}(x, x), Q_{n2}(x, x), \dots$. We put $\log x = \max\{1, \log x\}$, $\log_2 x = \log \log x$, $x > 0$.

Some useful lemmas. The following “decoupling” lemma is due to Martikainen [15] (first part) and Chow and Lai [4] (second part).

LEMMA 1.1. *Let $(U_n), (W_n)$ be two sequences of r. v.’s such that $U_n + W_n \rightarrow 0$ a. s. Suppose that one of the following conditions is satisfied:*

- 1) $(U_1, \dots, U_n, W_n) \stackrel{d}{=} (U_1, \dots, U_n, -W_n)$ for every n ;
- 2) (U_1, \dots, U_n) and W_n are independent for every n and $W_n \xrightarrow{\mathbf{P}} 0$.

Then $W_n \rightarrow 0$ a. s.

We frequently use the conditional versions of the Borel–Cantelli lemma and of the 3-series theorem (e.g., [26]).

LEMMA 1.2. *Let (B_i) be a sequence of events and (\mathcal{F}_i) be an increasing sequence of σ -fields, $\mathcal{F}_i \subset \mathcal{F}$, such that $B_i \in \mathcal{F}_i$ for each $i \geq 1$. Then $[B_i \text{ i. o.}]^C = [\sum_{i=1}^{\infty} \mathbf{P}(B_i | \mathcal{F}_{i-1}) < \infty]$. (Here i. o. means “infinitely often,” $A = B$ stands for $\mathbf{P}(A \Delta B) = 0$.)*

LEMMA 1.3. *Let $\mathcal{F}_1 \subset \mathcal{F}$ be an increasing sequence of σ -fields, Z_i be \mathcal{F}_i -measurable r. v.’s for each $i \geq 1$, C be a positive constant. Then the sums $Z_1 + \dots + Z_n$ converge a. s. on the event, where $\sum_{i=1}^{\infty} \mathbf{P}(|Z_i| > C | \mathcal{F}_{i-1}) < \infty$, $\sum_{i=1}^{\infty} \mathbf{E}(Z_i I(|Z_i| \leq C) | \mathcal{F}_{i-1})$ converges, $\sum_{i=1}^{\infty} \text{Var}(Z_i I(|Z_i| \leq C) | \mathcal{F}_{i-1}) < \infty$.*

Domains of partial attraction and p -stable random variables. We recall the definitions of certain classes of r. v.’s and their properties. Let Z, Z_1, Z_2, \dots be i. i. d. r. v.’s. We say that Z (or its distribution) belongs to the domain of partial attraction of the normal law (in short, $Z \in \text{DPA}(\mathcal{N})$) if there exist real numbers a_k, b_k and integers n_k such that $a_k(Z_1 + \dots + Z_{n_k} - b_k) \xrightarrow{d} \mathcal{N}(0, 1)$. If this relation does not hold for all sequences a_k, b_k, n_k we write $Z \notin \text{DPA}(\mathcal{N})$. For this definition and further properties we refer to Gnedenko and Kolmogorov [8]. They also give the following characterization property.

LEMMA 1.4. *$Z \notin \text{DPA}(\mathcal{N})$ if and only if $\mathbf{E}(Z^2 I(|Z| \leq x)) \leq Cx^2 \mathbf{P}(|Z| > x)$, $x \geq x_0$, for positive constants C and x_0 .*

We say that Z is p -stable with exponent $p \leq 2$ if for arbitrary real a, b with $a^2 + b^2 > 0$ there exists a constant c such that $(|a|^p + |b|^p)^{-1/p} (aZ_1 + bZ_2) \stackrel{d}{=} Z + c$. The p -stable distributions are known to be the only limit distributions for the suitable normalized sums $Z_1 + \dots + Z_n$. If $p = 2$ the r. v. Z is necessarily Gaussian. General references to p -stable distributions and their properties are in the books of Ibragimov and Linnik [9] and Feller [5]. There we also find the following statements.

LEMMA 1.5. Suppose that Z is p -stable with $p < 2$. Then

$$\lim_{x \rightarrow \infty} x^p \mathbf{P}(|Z| > x) = c_1, \quad \lim_{x \rightarrow \infty} \frac{x^2 \mathbf{P}(|Z| > x)}{\mathbf{E} Z^2 I(|Z| \leq x)} = c_2$$

for positive constants c_1, c_2 . Consequently, $Z \notin \text{DPA}(\mathcal{N})$.

Some facts on quadratic forms. Let $A_n = (a_{ij}^{(n)})_{i,j=1,\dots,n}$, $n = 1, 2, \dots$, be a sequence of real symmetric matrices. Put $Q_n(\varepsilon) = \sum_{i=1}^n a_{ii}^{(n)} + 2 \sum_{i=2}^n \sum_{j=1}^{i-1} a_{ij}^{(n)} \varepsilon_i \varepsilon_j$, where (ε_n) is a Rademacher sequence, i. e., the ε_n are i. i. d. with $\mathbf{P}(\varepsilon_1 = \pm 1) = \frac{1}{2}$.

LEMMA 1.6. If $Q_n(\varepsilon) \xrightarrow{\mathbf{P}} 0$ then $\lim \sum_{i=1}^n a_{ii}^{(n)} = 0$.

Proof. Without loss of generality we assume that $\sum_{i=1}^n (a_{ii}^{(n)})^2 \neq 0$ for all n . By Lemma 1.1 and Lemma 2.2 (i) and (iv) in [10], we infer that there exists a constant $\delta > 0$ such that $\mathbf{P}(|Q_n(\varepsilon)| > \delta (\mathbf{E} Q_n^2(\varepsilon))^{1/2}) > \delta$, but $\mathbf{P}(|Q_n(\varepsilon)| > \gamma) \rightarrow 0$, $\forall \gamma > 0$, so that

$$\mathbf{E} Q_n^2(\varepsilon) = \left(\sum_{i=1}^n a_{ii}^{(n)} \right)^2 + 4 \sum_{i=2}^n \sum_{j=1}^{i-1} (a_{ij}^{(n)})^2 \rightarrow 0.$$

The proof is completed.

For (X_n) i. i. d. $\mathcal{N}(0, 1)$ we make frequent use of the well-known identity

$$(1.1) \quad \widehat{Q}_n = \sum_{i=1}^n \sum_{j=1}^n a_{ij}^{(n)} X_i X_j \stackrel{d}{=} \sum_{i=1}^n \lambda_i^{(n)} X_i^2,$$

where $\lambda_i^{(n)}$ are the eigenvalues of A_n . By

$$\text{sp}(A_n) = \max_{i=1,\dots,n} |\lambda_i^{(n)}|$$

and by $\|A_n\| = (\sum_{i=1}^n \sum_{j=1}^n (a_{ij}^{(n)})^2)^{1/2}$ we denote the spectral and the Frobenius norms of A_n , respectively. For further theory concerning matrices, matrix norms, and eigenvalues we refer to Gantmacher [7] and Lancaster [12].

LEMMA 1.7. The following statements hold:

- 1)
$$\text{Var } \widehat{Q}_n = 2 \|A_n\|^2 = 2 \sum_{i=1}^n (\lambda_i^{(n)})^2,$$

$$\mathbf{E} \widehat{Q}_n = \text{tr}(A_n) = \sum_{i=1}^n a_{ii}^{(n)} = \sum_{i=1}^n \lambda_i^{(n)};$$
- 2)
$$\lim \frac{\text{sp}(A_n)}{\|A_n\|} = 0 \quad \text{if and only if} \quad \lim \frac{\|A_n^2\|}{\|A_n\|^2} = 0,$$

$$\liminf \frac{\text{sp}(A_n)}{\|A_n\|} > 0 \quad \text{if and only if} \quad \liminf \frac{\|A_n^2\|}{\|A_n\|^2} > 0.$$

The relations 1) are easily verified and 2) follows from the inequality $\text{sp}^2(A_n) \leq \|A_n^2\| = (\sum_{i=1}^n (\lambda_i^{(n)})^4)^{1/2} \leq \text{sp}(A_n) \|A_n\|$.

Next we give exponential tail estimates for the q. f.'s \widehat{Q}_n in i. i. d. $\mathcal{N}(0, 1)$ r. v.'s. The proofs are essentially the same as in [17], [19].

LEMMA 1.8. Suppose that $x_n \rightarrow \infty$. Choose $\delta \in (0, 1)$.

1) If $\lim[x_n \text{sp}(A_n)/\|A_n\|] = 0$ then for $n \geq n_0(\delta)$

$$\frac{x_n^2}{2} (1 - \delta) \leq -\log \mathbf{P}(|\hat{Q}_n - \mathbf{E}\hat{Q}_n| > x_n (\text{Var } \hat{Q}_n)^{1/2}) \leq \frac{x_n^2}{2} (1 + \delta).$$

2) If $\liminf \text{sp}(A_n)/\|A_n\| > 0$ then for $n \geq n_0(\delta)$

$$\frac{x_n}{2} (1 - \delta) \leq -\log \mathbf{P}(|\hat{Q}_n - \mathbf{E}\hat{Q}_n| > x_n \text{sp}(A_n)) \leq \frac{x_n}{2} (1 + \delta).$$

3) There exists a positive constant $C = C(\delta)$ such that

$$\mathbf{P}(|\hat{Q}_n - \mathbf{E}\hat{Q}_n| > x (\text{Var } \hat{Q}_n)^{1/2}) \leq C \exp\{-(1 - \delta)x/\sqrt{2}\}, \quad x > 0.$$

2. The independent symmetric case. Throughout this section (X_n) is a sequence of independent symmetric r. v.'s. We introduce six conditions:

- (1) the series $\overline{Q_{ni}}$ converges a. s., $i = 1, 2$;
- (2) the series $\overline{Q_n}$ converges a. s.;
- (3) $\lim b_n^{-1} Q_{ni} = 0$ a. s., $i = 1, 2$;
- (4) $\lim b_n^{-1} Q_n = 0$ a. s.;
- (5) $\lim b_n^{-1} a_{nn} X_n^2 = \lim b_n^{-1} X_n V_n(X) = 0$ a. s.;
- (6) $\lim b_n^{-1} (a_{nn} X_n^2 + 2X_n V_n(X)) = 0$ a. s.

THEOREM 2.1. For a sequence (X_n) of independent symmetric r. v.'s the following implications hold: $(i) \Rightarrow (j)$ provided that $i < j$, $(1) \Leftrightarrow (2)$, $(3) \Leftrightarrow (4)$, $(5) \Leftrightarrow (6)$.

Proof. The implications $(1) \Rightarrow (3) \Rightarrow (5)$, $(3) \Rightarrow (4)$ and $(5) \Rightarrow (6)$ are trivial. The implication $(1) \Leftrightarrow (2)$ is due to Krakowiak and Szulga [10, Theorem 3.6].

We show $(4) \Rightarrow (3)$. We suppose that (ε_n) is a Rademacher sequence independent of (X_n) . Then the sequences $(\varepsilon_n X_n)$ and (X_n) are identically distributed. We suppose that (ε_n) and (X_n) are defined on $[\Omega_1, \mathcal{F}_1, \mathbf{P}_1]$ and $[\Omega_2, \mathcal{F}_2, \mathbf{P}_2]$, respectively, and that $(\varepsilon_n X_n)$ is given on the product space. Let \mathbf{E}_i denote expectation with respect to \mathbf{P}_i . Put $Q_n(\varepsilon X) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} \varepsilon_i \varepsilon_j X_i X_j$. If $\lim b_n^{-1} Q_n(\varepsilon X) = 0$ a. s., then

$$\begin{aligned} 0 &= \lim_{k \rightarrow \infty} \mathbf{E}_2 \mathbf{P}_1 \left(\sup_{n \geq k} |b_n^{-1} Q_n(\varepsilon X)| > \gamma \right) \\ &= \mathbf{E}_2 \lim_{k \rightarrow \infty} \mathbf{P}_1 \left(\sup_{n \geq k} |b_n^{-1} Q_n(\varepsilon X)| > \gamma \right), \end{aligned}$$

$\forall \gamma > 0$, so that for \mathbf{P}_2 -almost all $\omega_2 \in \Omega_2$,

$$\lim b_n^{-1} \left(\sum_{i=1}^n a_{ii} X_i^2(\omega_2) + 2 \sum_{i=2}^n \sum_{j=1}^{i-1} a_{ij} \varepsilon_i(\omega_1) \varepsilon_j(\omega_1) X_i(\omega_2) X_j(\omega_2) \right) = 0$$

\mathbf{P}_1 -a. s. An application of Lemma 1.6 yields that $\lim b_n^{-1} \sum_{i=1}^n a_{ii} X_i^2 = 0$ a. s.

The proof of $(6) \Rightarrow (5)$ is similar and therefore omitted. Theorem 2.1 is proved.

The converse implications, i. e., $(j) \Rightarrow (i)$, $i < j$, are not true in general.

Example 2.2. Let (X_n) be a sequence of i. i. d. r. v.'s with $\mathbf{E}X_1 = 0$, $\mathbf{E}X_1^2 = 1$. Then $\lim n^{-1} X_n^2 = 0$ a. s., but $\lim n^{-1} \sum_{i=1}^n X_i^2 = 1$ a. s. and $\sum_{i=1}^{\infty} i^{-1} X_i^2 = \infty$ a. s., i. e., (5) does not imply (1) or (3), in general. Moreover, $\lim 2^{-n} n^{-1} \sum_{i=1}^n 2^i X_{2^i}^2 = 0$ a. s., but $\sum_{i=1}^{\infty} i^{-1} X_{2^i}^2 = \infty$ a. s., i. e., (3) does not imply (1) in general.

We introduce six analogous conditions for b. f.'s:

- (1') The series $\overline{Q_{ni}}(X, Y)$ converges a. s., $i = 1, 2$;
 (2') The series $\overline{Q_n}(X, Y)$ converges a. s.;
 (3') $\lim b_n^{-1} Q_{ni}(X, Y) = 0$ a. s., $i = 1, 2$;
 (4') $\lim b_n^{-1} Q_n(X, Y) = 0$ a. s.;
 (5') $\lim b_n^{-1} a_{nn} X_n Y_n = \lim b_n^{-1} (X_n V_n(Y) + Y_n V_n(X)) = 0$ a. s.;
 (6') $\lim b_n^{-1} (a_{nn} X_n Y_n + X_n V_n(Y) + Y_n V_n(X)) = 0$ a. s.

We get a result analogous to Theorem 2.1.

THEOREM 2.3. *Let (X_n) be a sequence of independent symmetric r. v. 's and (Y_n) be an independent copy of it. Then the following implications hold: $(i') \Rightarrow (j')$ provided that $i < j$, $(1') \Leftrightarrow (2')$, $(3') \Leftrightarrow (4')$, $(5') \Leftrightarrow (6')$.*

Proof. We show $(2') \Leftrightarrow (1')$. The remaining statements can be proved analogously to the corresponding parts in the proof of Theorem 2.1.

From $(2')$ we get $\overline{Q_n}(X, Y) - \overline{Q_m}(X, Y) \rightarrow 0$ a. s., as $m, n \rightarrow \infty$.

Let (ε_n) be a Rademacher sequence independent of (X_n) and (Y_n) . We suppose that (ε_n) and (X_n, Y_n) are defined on $[\Omega_1, \mathcal{F}_1, \mathbf{P}_1]$ and $[\Omega_2, \mathcal{F}_2, \mathbf{P}_2]$, respectively. Then

$$\begin{aligned}
 0 &= \lim_{k \rightarrow \infty} \mathbf{E}_2 \mathbf{P}_1 \left(\sup_{n \geq m \geq k} |\overline{Q_n}(\varepsilon X, \varepsilon Y) - \overline{Q_m}(\varepsilon X, \varepsilon Y)| > \gamma \right) \\
 &= \mathbf{E}_2 \lim_{k \rightarrow \infty} \mathbf{P}_1 \left(\sup_{n \geq m \geq k} |\overline{Q_n}(\varepsilon X, \varepsilon Y) - \overline{Q_m}(\varepsilon X, \varepsilon Y)| > \gamma \right),
 \end{aligned}$$

$\forall \gamma > 0$, so that for \mathbf{P}_2 -almost all $\omega_2 \in \Omega_2$,

$$\begin{aligned}
 \overline{Q_n}(\varepsilon X, \varepsilon Y) - \overline{Q_m}(\varepsilon X, \varepsilon Y) &= \sum_{i=m+1}^n a_{ii} X_i(\omega_2) Y_i(\omega_2) \\
 &+ \sum_{i=m+1}^n \sum_{j=1}^m a_{ij} \varepsilon_i(\omega_1) \varepsilon_j(\omega_1) (X_i(\omega_2) Y_j(\omega_2) + Y_i(\omega_2) X_j(\omega_2)) \\
 &+ \sum_{i=m+1}^n \sum_{j=m+1}^{i-1} a_{ij} \varepsilon_i(\omega_1) \varepsilon_j(\omega_1) (X_i(\omega_2) Y_j(\omega_2) + Y_i(\omega_2) X_j(\omega_2)) \rightarrow 0
 \end{aligned}$$

\mathbf{P}_1 -a. s. An application of Lemma 1.6 yields that $(1')$ holds.

Example 2.4. Let (X_n) be i. i. d. symmetric r. v. 's with $\mathbf{E} X_1^2 = 1$. Then

$$\lim (n \log_2 n)^{-1} \left(X_n \sum_{j=1}^{n-1} Y_j + Y_n \sum_{j=1}^{n-1} X_j \right) = 0$$

a. s., but

$$\begin{aligned}
 &\limsup (n \log_2 n)^{-1} \sum_{i=2}^n \left(X_i \sum_{j=1}^{i-1} Y_j + Y_i \sum_{j=1}^{i-1} X_j \right) \\
 &= \limsup (n \log_2 n)^{-1} \left(\sum_{i=1}^n X_i \sum_{j=1}^n Y_j - \sum_{i=1}^n X_i Y_i \right) = 1 \quad \text{a. s.}
 \end{aligned}$$

Hence $(5')$ does not imply $(3')$ in general.

For the same example, assume that $(1')$ is satisfied. But then the Kronecker lemma would yield $(3')$. Hence $(5')$ does not imply $(1')$ in general.

Moreover, $\lim n^{-1/2}(\log_2 n)^{-1} \sum_{i=1}^n X_i Y_i = 0$ a.s., but $\sum_{i=1}^\infty i^{-1/2}(\log_2 i)^{-1} X_i Y_i$ does not converge a.s., i.e., (3') does not imply (1') in general.

Remark. Clearly, it is more convenient to deal with the b.f.'s $Q_n(X, Y)$ than with the q.f.'s Q_n , because such b.f.'s contain "less" dependence than q.f.'s.

Kwapień and Woyczynski [11] showed that the q.f.'s Q_{n2} converge a.s. if and only if the b.f.'s $Q_{n2}(X, Y)$ do.

The results in §§ 3 and 4 state that the SLLN's

$$\lim b_n^{-1} Q_{n2} = 0 \quad \text{a.s. and} \quad \lim b_n^{-1} Q_{n2}(X, Y) = 0 \quad \text{a.s.}$$

hold at the same time provided that (X_n) is a sequence of i.i.d. symmetric p -stable r.v.'s with $p \leq 2$. We conjecture that the SLLN's for Q_{n2} and $Q_{n2}(X, Y)$ are equivalent in the general (symmetric) case.

3. The non-Gaussian case. In this section we show the equivalence of the conditions (1)–(6) and (1')–(6') (cf. § 2) for i.i.d. r.v.'s X_n outside the domain of partial attraction of the normal law, i.e., $X_1 \notin \text{DPA}(\mathcal{N})$ (cf. § 2).

PROPOSITION 3.1. *For sequences (X_n) and (Y_n) of i.i.d. symmetric r.v.'s with $X_1 \notin \text{DPA}(\mathcal{N})$ the conditions (1)–(6) and (1')–(6') are equivalent. Moreover, if $a_{ii} = 0$ for all i then (1) and (1') are equivalent.*

Proof. We show the equivalence of (1)–(6). By Theorem 2.1 it suffices to show the implication (5) \Rightarrow (1). The relation $\lim b_n^{-1} a_{nn} X_n^2 = 0$ a.s., the Borel–Cantelli lemma, and Lemma 1.4 imply that the assumptions of the 3-series theorem are satisfied for the sequence $b_n^{-1} a_{nn} X_n^2$. Hence $\overline{Q_{n1}}$ converges a.s. Similarly, the relation $\lim b_n^{-1} X_n V_n(X) = 0$ a.s. and Lemma 1.2 imply that

$$\sum_{n=1}^{\infty} \mathbf{P}\left(|X_n V_n(X)| > b_n \mid \mathcal{F}_{n-1}\right) < \infty \quad \text{a.s.},$$

where \mathcal{F}_n is the σ -field generated by X_1, \dots, X_n . Applying Lemma 1.4 and using the symmetry of X_n we get

$$\begin{aligned} \sum_{n=1}^{\infty} b_n^{-2} \mathbf{E}\left((X_n V_n(X))^2 I(|X_n V_n(X)| \leq b_n) \mid \mathcal{F}_{n-1}\right) &< \infty \quad \text{a.s.}, \\ \sum_{n=1}^{\infty} b_n^{-1} \mathbf{E}\left(X_n V_n(X) I(|X_n V_n(X)| \leq b_n) \mid \mathcal{F}_{n-1}\right) &= 0 \quad \text{a.s.} \end{aligned}$$

Lemma 1.3 yields that $\overline{Q_{n2}}$ converges a.s.

Next we show the equivalence of (1')–(6'). By Theorem 2.3 it suffices to show the implication (5') \Rightarrow (1'). It is not difficult to see that $X_1 Y_1 \notin \text{DPA}(\mathcal{N})$. Indeed, by Lemma 1.4,

$$\begin{aligned} \mathbf{E}(X_1 Y_1)^2 I(|X_1 Y_1| \leq x) &= \mathbf{E}(X_1)^2 \mathbf{E}(Y_1^2 I(|X_1 Y_1| \leq x) \mid X_1) \\ &\leq C x^2 \mathbf{P}(|X_1 Y_1| > x), \quad x > x_0. \end{aligned}$$

Put $U_n = b_n^{-1} X_n V_n(Y)$, $W_n = b_n^{-1} Y_n V_n(X)$. For a Borel set A we have

$$\begin{aligned} \mathbf{P}((U_1, \dots, U_n, W_n) \in A) &= \mathbf{E} \mathbf{P}\left((U_1, \dots, U_n, -b_n^{-1} Y_n V_n(X)) \right. \\ &\quad \left. \in A \mid \mathcal{F}_n, \mathcal{G}_{n-1}\right) = \mathbf{P}((U_1, \dots, U_n, -W_n) \in A), \end{aligned}$$

where \mathcal{G}_n is the σ -field generated by Y_1, \dots, Y_n .

By (5'), $U_n + W_n \rightarrow 0$ a.s. An application of Lemma 1.1 yields that $\lim b_n^{-1} X_n \times V_n(Y) = 0$ a.s.

Using similar arguments as in the first part of the proof, it is not difficult to see that the series $\sum b_n^{-1} a_{nn} X_n Y_n$, $\sum b_n^{-1} X_n V_n(Y)$, $\sum b_n^{-1} Y_n V_n(X)$ converge a.s., thus implying (1').

Now suppose that $a_{ii} = 0$ for all i . To prove $(1) \iff (1')$ it suffices to show $(5) \iff (5')$. From (5) we get, by Lemma 1.2, that

$$\begin{aligned} \infty &> \sum_{n=1}^{\infty} \mathbf{P}\left(|X_n V_n(X)| > \varepsilon b_n \mid \mathcal{F}_{n-1}\right) \\ &= \sum_{n=1}^{\infty} \mathbf{P}\left(|Y_n V_n(X)| > \varepsilon b_n \mid \mathcal{F}_{n-1}, \mathcal{G}_{n-1}\right) \quad \text{a.s., } \forall \varepsilon > 0. \end{aligned}$$

Hence $\lim b_n^{-1} Y_n V_n(X) = 0$ a.s. This implies (5'). The converse implication follows analogously. Proposition 3.1 is proved.

Remarks. 1) The subject is much more complicated if the X_n are nonsymmetric r.v.'s. For the limit theory of sums of independent r.v.'s there exist well-known symmetrization and desymmetrization techniques. In the case of q.f.'s such general methods are not known to the author.

2) From Lemma 1.5 we get immediately that if X_1 is symmetric and p -stable with $p < 2$, the assumptions of Proposition 3.1 are satisfied. In Proposition 3.2 we consider certain classes of nonsymmetric p -stable r.v.'s for which the statement of Proposition 3.1 remains valid.

The papers of Cambanis et al. [3], Krakowiak and Szulga [10], and Kwapien and Woyczynski [11] show that the a.s. convergence of the q.f.'s Q_n in independent symmetric r.v.'s can be described by certain nonrandom characteristics. In Proposition 3.2 we use this idea for the SLLN.

PROPOSITION 3.2. *Let (X_n) be a sequence of i.i.d. p -stable r.v.'s. Suppose that one of the following conditions is satisfied:*

- a) $p \leq 1$ and $\limsup_{x \rightarrow \infty} |\mathbf{E} X_1 I(|X_1| \leq x)| < \infty$;
- b) $0 < p < 2$ and $\mathbf{E} X_1 = 0$.

Then the conditions (1)–(6) are equivalent, and (1) holds if and only if

$$(3.1a) \quad \sum_{n=1}^{\infty} |b_n^{-1} a_{nn}|^{p/2} < \infty,$$

$$(3.1b) \quad \sum_{i=2}^{\infty} \sum_{j=1}^{i-1} |b_i^{-1} a_{ij}|^p \left(1 + \log \frac{|b_i^{-1} a_{ij}|^p}{\sum_{l=i+1}^{\infty} |b_l^{-1} a_{il}|^p \sum_{l=1}^{j-1} |b_j^{-1} a_{lj}|^p} \right) < \infty.$$

Proof. The implications $(1) \Rightarrow (3) \Rightarrow (5)$, $(1) \Rightarrow (2) \Rightarrow (4) \Rightarrow (6)$ are trivial. We show $(6) \Rightarrow (5)$.

Let \mathcal{F}_n and \mathcal{G}_n be the σ -fields generated by X_1, \dots, X_n and Y_1, \dots, Y_n , respectively. From (6) and Lemma 1.2 we conclude that

$$\begin{aligned} \infty &> \sum \mathbf{P}\left(|a_{nn} X_n^2 + 2X_n V_n(X)| > \varepsilon b_n \mid \mathcal{F}_{n-1}\right) \\ &= \sum \mathbf{P}\left(|a_{nn} Y_n^2 + 2Y_n V_n(X)| > \varepsilon b_n \mid \mathcal{F}_{n-1}, \mathcal{G}_{n-1}\right) \quad \text{a.s., } \forall \varepsilon > 0. \end{aligned}$$

Hence $\lim b_n^{-1}(a_{nn}Y_n^2 + 2Y_nV_n(X)) = 0$ a.s., $\lim b_n^{-1}Y_nV_n(X - Y) = 0$ a.s. Using Lemma 1.2 again and recalling the tail behavior of p -stable r.v.'s (see Lemma 1.5), we get

$$(3.2) \quad \sum \left(b_n^{-1} |V_n(X - Y)| \right)^p < \infty \quad \text{a.s.}$$

Let $\|\cdot\|_p$ denote the usual l^p -quasinorm. Put

$$u_j = (\underbrace{0, \dots, 0}_{j \text{ times}}, b_{j+1}^{-1}a_{j,j+1}, b_{j+2}^{-1}a_{j,j+2}, \dots),$$

$U_j = u_j X_j$, $\hat{U}_j = u_j Y_j$, $Z^1 = Z I(\|Z\|_p \leq 1)$ for any random element Z taking values in l^p . Following the lines of the proof of Theorem 2.2 in [3] we see that (3.2) yields the a.s. convergence of $\sum (U_j - \hat{U}_j)$ in l^p .

We show that $\sum U_j$ converges a.s.

By the Borel–Cantelli lemma, $\sum \mathbf{P}(\|U_j\|_p > 1) < \infty$, so that $\sum (U_j^1 - \hat{U}_j^1)$ converges a.s. By a desymmetrization argument, there exist constants $y_j \in l^p$ with $\|y_j\|_p \leq 1$, such that $\sum (U_j^1 - y_j)$ converges a.s. From Theorem 3.7.1 in [2] we conclude that $\mathbf{E} \sup_n \|\sum_{j=1}^n (U_j^1 - y_j)\|_p < \infty$. Now, Lebesgue's theorem implies that $\sum \mathbf{E}(U_j^1 - y_j)$ converges in l^p . Here $\mathbf{E}U_j^1$ exists as a Pettis integral in l^p and $\mathbf{E}U_j^1 = u_j \mathbf{E}X_j I(\|U_j\|_p \leq 1)$.

We have

$$\begin{aligned} \|\mathbf{E}U_j^1\|_p &= \|u_j\|_p \left| \mathbf{E}X_1 I(\|u_j\|_p |X_1|^p < 1) \right|^p \leq \text{const } \|u_j\|_p \\ &\leq \text{const } \mathbf{P}(\|U_j\|_p > 1), \quad p \leq 1, \\ \|\mathbf{E}U_j^1\|_p &= \|u_j\|_p \left| \mathbf{E}X_1 I(\|U_j\|_p > 1) \right| \leq \text{const } \mathbf{P}(\|U_j\|_p > 1), \quad 1 < p < 2. \end{aligned}$$

Hence $\|\sum_{j=m}^n \mathbf{E}U_j^1\|_p \leq \text{const } \sum_{j=m}^n \mathbf{P}(\|U_j\|_p > 1)$ thus implying the convergence of the series $\sum \mathbf{E}U_j^1$ and $\sum y_j$ in l^p . Hence $\sum U_j$ converges a.s. in l^p and again using the arguments in the proof of Theorem 2.2 in [3] we get

$$(3.3) \quad \sum |b_n^{-1}V_n(X)|^p < \infty \quad \text{a.s.}$$

The proof of the implication (5) \Rightarrow (1) is similar to the general symmetric case (see Proposition 3.1). We omit it.

It remains to show that (5) and (3.1) are equivalent. The relation $\lim b_n^{-1}a_{nn}X_n^2 = 0$ a.s. holds if and only if $\sum \mathbf{P}(|a_{nn}X_n^2| > b_n) < \infty$. Recalling the tail behavior of X_n^2 (see Lemma 1.5) we conclude that the latter condition and (3.1a) are equivalent.

From the first part of the proof we infer that (3.2) holds if and only if (3.3) is satisfied. Hence $\lim b_n^{-1}X_nV_n(X) = 0$ a.s. and (3.2) are equivalent, and in [3] the equivalence of (3.2) and (3.1b) is shown.

4. The Gaussian case. Throughout (X_n) is a sequence of i.i.d. $\mathcal{N}(0, 1)$ r.v.'s. Example 2.2 shows that in this case the conditions (1)–(6) are not equivalent, but in view of Theorem 2.1 we have (1) \iff (2) and (3) \iff (4). So it is of interest to characterize (1) and (3) in dependence on the distribution of the X_n and the coefficients a_{ij} .

From [29] (see also [25]) it follows that (1) holds if and only if $\sum_{i=1}^{\infty} \sum_{j=1}^i b_i^{-2} a_{ij}^2 < \infty$ and $\sum_{i=1}^{\infty} b_i^{-1} a_{ii}$ converges.

For the SLLN $\lim b_n^{-1} Q_n = 0$ a.s. we want to give a Prokhorov-type characterization (cf. Introduction).

It should be noted that the SLLN's for Q_n and for $Q_n(X, Y)$ obey the 0–1 law for Gaussian q. f.'s proved by de Acosta [1].

We introduce some further notation:

$$n_k = \max\{n: b_n \leq 2^k\}, \quad \sum_k = \sum_{i=n_{k-1}+1}^{n_k}, \quad \sum_{k'} = \sum_{j=1}^{n_{k-1}},$$

$$\sum_{kk} = \sum_{i,j=n_{k-1}+1}^{n_k}, \quad \sum_{kk'} = \sum_{i=n_{k-1}+1}^{n_k} \sum_{j=1}^{n_{k-1}}.$$

The main result is the following theorem.

THEOREM 4.1. *The SLLN $\lim b_n^{-1} Q_n = 0$ a.s. holds if and only if*

$$(4.1) \quad \lim 2^{-k} \sum_{kk} a_{ij} X_i X_j = 0 \quad \text{a.s.},$$

$$(4.2) \quad \lim 2^{-k} \sum_{kk'} a_{ij} X_i X_j = 0 \quad \text{a.s.},$$

$$(4.3) \quad \lim b_n^{-1} \sum_{i=1}^n a_{ii} = 0.$$

The SLLN $\lim b_n^{-1} (Q_n - \mathbf{E}Q_n) = 0$ a.s. holds if and only if (4.2) and

$$(4.4) \quad \lim 2^{-k} \sum_{kk} a_{ij} (X_i X_j - \mathbf{E}X_i X_j) = 0 \quad \text{a.s.}$$

are satisfied.

Remarks. 1) The statements of Theorem 4.1 remain valid if (n_k) is defined by $n_k = \max\{n: b_n \leq d^k\}$ for some real $d > 1$ and if 2^{-k} is replaced by d^{-k} .

2) Theorem 4.1 means that the SLLN $\lim b_n^{-1} Q_n = 0$ a.s. holds if and only if $\lim b_{n_k}^{-1} (Q_{n_k} - Q_{n_{k-1}}) = 0$ a.s. and $\lim b_n^{-1} \mathbf{E}Q_n = 0$.

Applying Lemma 1.2 to (4.2) we get that this relation holds if and only if $\lim 2^{-2k} Y_k^2 \sum_k (\sum_{k'} a_{ij} X_j)^2 = 0$ a.s. From this fact and the Borel–Cantelli lemma we get the following corollary.

COROLLARY 4.2. *The SLLN $\lim b_n^{-1} Q_n = 0$ a.s. holds if and only if the conditions (4.3) and*

$$(4.5) \quad \sum_{k=1}^{\infty} \mathbf{P} \left(\left| \sum_{kk} a_{ij} (X_i X_j - \mathbf{E}X_i X_j) \right| > \varepsilon 2^k \right) < \infty, \quad \forall \varepsilon > 0,$$

$$(4.6) \quad \sum_{k=1}^{\infty} \mathbf{P} \left(1 - \Phi \left(\frac{\varepsilon 2^k}{(\sum_k (\sum_{k'} a_{ij} X_j)^2)^{1/2}} \right) \right) < \infty \quad \text{a.s.}, \quad \forall \varepsilon > 0,$$

are satisfied. Here $\Phi(\cdot)$ denotes the distribution function of X_1 .

The SLLN $\lim b_n^{-1} (Q_n - \mathbf{E}Q_n) = 0$ a.s. holds if and only if the conditions (4.5) and (4.6) are satisfied.

Next we want to discuss the conditions (4.5) and (4.6).

Condition (4.6). It is satisfied if $\sum_k (\sum_{k'} a_{ij} X_j)^2 = o(2^{2k} / \log k)$ a.s., or, equivalently, if

$$(4.7) \quad T_k = \sum_{j,l=1}^{n_{k-1}} X_l X_j \sum_k a_{li} a_{ij} = o \left(\frac{b_{n_k}^2}{\log b_{n_k}} \right) \quad \text{a.s.}$$

Note that $o(\cdot)$ may not be replaced by $O(\cdot)$.

Using Lemma 1.8, 3) and a Borel–Cantelli argument, we see that (4.7) holds if

$$(4.8) \quad \text{Var } T_k = 2 \sum_{j,l=1}^{n_k-1} \left(\sum_k a_{lj} a_{ij} \right)^2 = o\left(\frac{b_{n_k}^4}{\log^4 b_{n_k}}\right),$$

$$(4.9) \quad \sum_{k'} \sum_k a_{ij}^2 = o\left(\frac{b_{n_k}^2}{\log b_{n_k}}\right).$$

The condition

$$(4.10) \quad \sum_{k'} \sum_k a_{ij}^2 = o\left(\frac{b_{n_k}^2}{\log^2 b_{n_k}}\right)$$

implies both conditions (4.8) and (4.9). It is a consequence of the proofs in § 5 (see (5.2)) that $\text{Var } Q_n = 2 \sum_{i,j=1}^n a_{ij}^2 = o(b_n^2)$, so that (4.10) is a very mild condition.

Condition (4.5). In § 1 (cf. (1.1)) we mentioned that $\sum_{k,k} a_{ij}(X_i X_j - \mathbf{E} X_i X_j) \stackrel{d}{=} \sum_k \lambda_i^{(k)}(X_i^2 - 1)$, where $\lambda_i^{(k)}$ are the eigenvalues of $A_k = (a_{ij})$, $i, j = n_{k-1} + 1, \dots, n_k$.

In view of the Prokhorov-type characterization of the SLLN for sums of independent r.v.'s given by Martikainen [13] the condition (4.5) holds if and only if $\lim 2^{-k} \sum_{l=1}^k \sum_k \lambda_i^{(k)}(X_i^2 - 1) = 0$ a.s. From Martikainen's [14] paper it follows that the latter relation is equivalent to the convergence of certain nonrandom series depending on the distribution of X_1 and on the a_{ij} .

This approach is not very convenient because we need all eigenvalues of the matrices A_k .

Recalling the exponential tail estimates of Lemma 1.8 we get the following consequence of Theorem 4.1. Below we put $\frac{1}{0} = \infty$ and $e^{-\infty} = 0$.

COROLLARY 4.3. *Suppose that (4.6) is satisfied.*

1) *If $\liminf[\text{sp}(A_k)/\|A_k\|] > 0$, then the SLLN $\lim b_n^{-1}(Q_n - \mathbf{E}Q_n) = 0$ a.s. holds if and only if*

$$(4.11) \quad \sum \exp \left\{ \frac{-\varepsilon 2^k}{\|A_k\|} \right\} < \infty, \quad \forall \varepsilon > 0.$$

2) *If $\lim[2^k \text{sp}(A_k)/\|A_k\|^2] = 0$, then the SLLN $\lim b_n^{-1}(Q_n - \mathbf{E}Q_n) = 0$ a.s. holds if and only if $\sum \exp\{-\varepsilon 2^{2k}/\|A_k\|^2\} < \infty$, $\forall \varepsilon > 0$.*

3) *The condition (4.11) is sufficient for the SLLN $\lim b_n^{-1}(Q_n - \mathbf{E}Q_n) = 0$ a.s.*

For an interpretation of the conditions used in this corollary we refer to Lemma 1.7.

The proof of Theorem 4.1 makes essential use of the following fact.

THEOREM 4.4. *The SLLN $\lim b_n^{-1}Q_n = 0$ a.s. holds if and only if the condition (4.3) and the SLLN $\lim b_n^{-1}Q_n(X, Y) = 0$ a.s. are satisfied.*

The SLLN $\lim b_n^{-1}(Q_n - \mathbf{E}Q_n) = 0$ a.s. holds if and only if the SLLN

$$\lim b_n^{-1}Q_n(X, Y) = 0 \quad \text{a.s.}$$

is satisfied.

Theorems 4.1 and 4.4 are proved in § 5.

We also give a characterization of the SLLN $\lim b_n^{-1} Q_n(X, Y) = 0$ a. s. (cf. Lemma 5.3) which is similar to Theorem 4.4.

5. Proofs of Theorems 4.1 and 4.4. Throughout \mathcal{F}_n and \mathcal{G}_n denote the σ -fields generated by X_1, \dots, X_n and Y_1, \dots, Y_n , respectively.

For q. f.'s the so-called polarization equality holds, i. e.,

$$(5.1) \quad Q_n(X + Y, X - Y) = Q_n(X, X) - Q_n(Y, Y).$$

For i. i. d. $\mathcal{N}(0, 1)$ r. v.'s (X_n) and (Y_n) the sequences $X + Y = (X_n + Y_n)$ and $X - Y = (X_n - Y_n)$ are i. i. d., so that $\frac{1}{2} Q_n(X + Y, X - Y)$ and $Q_n(X, Y)$ are identically distributed.

Proof of Theorem 4.4. Suppose that $\lim b_n^{-1} Q_n = 0$ a. s. Then the SLLN $\lim b_n^{-1} Q_n(X, Y) = 0$ a. s. is an immediate consequence of (5.1) and the remark following it. Note that $\lim b_n^{-1} \text{med } Q_n = 0$ and $b_n^{-1} |\text{med } Q_n - \mathbf{E} Q_n| \leq b_n^{-1} (2 \text{Var } Q_n)^{1/2}$. We show that

$$(5.2) \quad b_n^{-2} \text{Var } Q_n \longrightarrow 0,$$

thus implying that $b_n^{-1} \mathbf{E} Q_n = b_n^{-1} \sum_{i=1}^n a_{ii} \rightarrow 0$. We have $\mathbf{P}(|Q_n(X, Y)| > \varepsilon b_n) \rightarrow 0$, $\forall \varepsilon > 0$, and

$$\begin{aligned} \mathbf{P}(|Q_n(X, Y)| > \varepsilon b_n) &= \mathbf{EP} \left(\left| \sum_{i=1}^n X_i \sum_{j=1}^n a_{ij} Y_j \right| > \varepsilon b_n \mid \mathcal{G}_n \right) \\ &= \mathbf{P} \left(X_1^2 \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij} Y_j \right)^2 > \varepsilon^2 b_n^2 \right) = \mathbf{P} \left(X_1^2 \sum_{i=1}^n \lambda_i^{(n)} Y_j^2 > \varepsilon^2 b_n^2 \right), \end{aligned}$$

where $\lambda_i^{(n)}$ are the eigenvalues of the matrix A_n^2 , $A_n = (a_{ij})$, $i, j = 1, \dots, n$ (cf. (1.1)). By the criterion on the weak law of large numbers for sums of independent r. v.'s (e. g., [23]),

$$b_n^{-2} \sum_{i=1}^n \lambda_i^{(n)} \mathbf{E}(X_1, Y_i)^2 I(\lambda_i^{(n)} (X_1 Y_i)^2 \leq b_n^2) \longrightarrow 0,$$

so that

$$b_n^{-2} \sum_{i=1}^n \lambda_i^{(n)} = b_n^{-2} \text{tr}(A_n^2) = b_n^{-2} \sum_{j,i=1}^n a_{ij}^2 = b_n^{-2} 2^{-1} \text{Var } Q_n \longrightarrow 0$$

(cf. Lemma 1.7).

Now suppose that $\lim b_n^{-1} Q_n(X, Y) = 0$ a. s. and $b_n^{-1} \sum_{i=1}^n a_{ii} \rightarrow 0$. From (5.1) we infer that

$$b_n^{-1} \left((Q_n(X, X) - \mathbf{E} Q_n(X, X)) - (Q_n(Y, Y) - \mathbf{E} Q_n(Y, Y)) \right) \longrightarrow 0 \quad \text{a. s.}$$

and, by (5.2), $b_n^{-1} (Q_n - \mathbf{E} Q_n) \xrightarrow{\mathbf{P}} 0$. An application of Lemma 1.1 yields that $b_n^{-1} (Q_n - \mathbf{E} Q_n) \longrightarrow 0$ a. s. Hence $b_n^{-1} Q_n \rightarrow 0$ a. s.

The proof of the remaining statement is similar. We omit it.

Proof of Theorem 4.1. In view of Theorem 4.4 we may consider $Q_n(X, Y)$ instead of Q_n . The proof is given by a sequence of lemmas.

LEMMA 5.1. $2^{-k} Q_{n_k}(X, Y) \rightarrow 0$ a. s. holds if and only if $2^{-k} (Q_{n_k}(X, Y) - Q_{n_{k-1}}(X, Y)) \longrightarrow 0$ a. s.

Proof. \Rightarrow : Trivial.

\Leftarrow : Note that

$$Q_{n_k}(X, Y) = \sum_{l=1}^k 2^l \left(2^{-1} (Q_{n_l}(X, Y) - Q_{n_{l-1}}(X, Y)) \right).$$

It remains to apply the Toeplitz lemma to $2^{-k} Q_{n_k}(X, Y)$.

LEMMA 5.2.

$$2^{-k} (Q_{n_k}(X, Y) - Q_{n_{k-1}}(X, Y)) \longrightarrow 0 \quad \text{a.s.}$$

if and only if

$$(5.3) \quad \lim 2^{-k} \sum_{kk'} a_{ij} X_i Y_j = 0 \quad \text{a.s.},$$

$$(5.4) \quad \lim 2^{-k} \sum_{kk'} a_{ij} X_i Y_j = 0 \quad \text{a.s.}$$

Proof. \Leftarrow : Trivial.

\Rightarrow : Put $U_k = 2^{-k} \sum_{kk'} a_{ij} X_i Y_j = 0$, $W_k = 2^{-k} \sum_{kk'} a_{ij} (X_i Y_j + Y_i X_j)$. Then $U_k + W_k \rightarrow 0$ a.s. For any Borel set A we have

$$\begin{aligned} & \mathbf{P}((U_1, \dots, U_k, W_k) \in A) \\ &= \mathbf{EP} \left((U_1, \dots, U_{k-1}, 2^{-k} \sum_{kk'} a_{ij} (-X_i)(-Y_j), 2^{-k} \sum_{kk'} a_{ij} (-X_i Y_j - Y_i X_j)) \right. \\ & \quad \left. \in A \mid \mathcal{F}_{n_{k-1}}, \mathcal{G}_{n_{k-1}} \right) = \mathbf{P}((U_1, \dots, U_k, -W_k) \in A). \end{aligned}$$

An application of Lemma 1.1 yields (5.3). In the same way one can show that $W_k \rightarrow 0$ a.s. implies (5.4). Lemma 5.2 is proved.

LEMMA 5.3. $\lim b_n^{-1} Q_n(X, Y) = 0$ a.s. holds if and only if $2^{-k} (Q_{n_k}(X, Y) - Q_{n_{k-1}}(X, Y)) \longrightarrow 0$ a.s.

Proof. \Rightarrow : Trivial.

\Leftarrow : Choose $n \in (n_{k-1}, n_k]$. Then

$$\begin{aligned} Q_{n_k}(X, Y) &= Q_{n_{k-1}}(X, Y) + \sum_{i,j=n_{k-1}+1}^n a_{ij} X_i Y_j \\ &+ \sum_{i=n_{k-1}+1}^n \sum_{k'} a_{ij} (X_i Y_j + Y_i X_j). \end{aligned} \quad (5.5)$$

By Lemma 5.1,

$$(5.6) \quad 2^{-k} Q_{n_{k-1}}(X, Y) \longrightarrow 0 \quad \text{a.s.}$$

In view of Lemma 5.2 we have (5.4). Lemma 1.2 implies that

$$(5.7) \quad \sum \mathbf{P} \left(\left| \sum_{kk'} a_{ij} X_i Y_j \right| > \varepsilon 2^k \mid \mathcal{F}_{n_{k-1}}, \mathcal{G}_{n_{k-2}} \right) < \infty \quad \text{a.s.},$$

$\forall \varepsilon > 0$. By Lévy's inequality we get

$$\begin{aligned} & \mathbf{P} \left(\max_{n_{k-1} < n \leq n_k} \left| \sum_{i=n_{k-1}+1}^n \sum_{k'} a_{ij} X_i Y_j \right| > \varepsilon 2^k \mid \mathcal{F}_{n_{k-1}}, \mathcal{G}_{n_{k-2}} \right) \\ &= \mathbf{EP} \left(\max_{n_{k-1} < n \leq n_k} \left| \sum_{i=n_{k-1}+1}^n \sum_{k'} a_{ij} X_i Y_j \right| > \varepsilon 2^k \mid \mathcal{F}_{n_{k-1}}, \mathcal{G}_{n_{k-1}} \mid \mathcal{F}_{n_{k-1}}, \mathcal{G}_{n_{k-2}} \right) \\ &\leq 2\mathbf{P} \left(\left| \sum_{kk'} a_{ij} X_i Y_j \right| > \varepsilon 2^k \mid \mathcal{F}_{n_{k-1}}, \mathcal{G}_{n_{k-2}} \right). \end{aligned}$$

This and (5.7) imply that

$$(5.8) \quad 2^{-k} \sum_{i=n_{k-1}+1}^n \sum_{k'} a_{ij} X_i Y_j \longrightarrow 0 \quad \text{a. s.}$$

uniformly for $n \in (n_{k-1}, n_k]$.

We have

$$(5.9) \quad \sum_{i,j=n_{k-1}+1}^n a_{ij} X_i Y_j = \sum_{i=n_{k-1}+1}^n X_i \left(\sum_{j=n_{k-1}+1}^{n_k} a_{ij} Y_j - \sum_{j=n+1}^{n_k} a_{ij} Y_j \right).$$

From Lemma 5.2 we get (5.3). Hence

$$\begin{aligned} & \sum_k \mathbf{P} \left(\max_{n_{k-1} < n \leq n_k} \left| \sum_{i=n_{k-1}+1}^n X_i \sum_{j=n_{k-1}+1}^{n_k} a_{ij} Y_j \right| > \varepsilon 2^k \right) \\ &= \sum_k \mathbf{EP} \left(\max_{n_{k-1} < n \leq n_k} \left| \sum_{i=n_{k-1}+1}^n X_i \sum_{j=n_{k-1}+1}^{n_k} a_{ij} Y_j \right| > \varepsilon 2^k \mid \mathcal{G}_{n_k} \right) \\ &\leq 2 \sum_k \mathbf{P} \left(\left| \sum_{kk} a_{ij} X_i Y_j \right| > \varepsilon 2^k \right) < \infty, \quad \forall \varepsilon > 0. \end{aligned}$$

This implies that

$$\begin{aligned} & 2^{-k} \sum_{i=n_{k-1}+1}^n X_i \sum_{j=n_{k-1}+1}^{n_k} a_{ij} Y_j \longrightarrow 0 \quad \text{a. s.}, \\ (5.10) \quad & 2^{-k} \sum_{i=n_{k-1}+1}^n \sum_{j=n_{k-1}+1}^{n_k} a_{ij} (X_i Y_j - Y_i X_j) \\ &= 2^{-k} \sum_{i=n_{k-1}+1}^n \sum_{j=n+1}^{n_k} a_{ij} (X_i Y_j - Y_i X_j) \longrightarrow 0 \quad \text{a. s.} \end{aligned}$$

uniformly for $n \in (n_{k-1}, n_k]$.

Applying Lemma 1.1 we conclude that

$$(5.11) \quad 2^{-k} \sum_{i=n_{k-1}+1}^n \sum_{j=n+1}^{n_k} a_{ij} X_i Y_j \longrightarrow 0 \quad \text{a. s.}$$

Combining (5.9)–(5.11) we obtain

$$(5.12) \quad 2^{-k} \sum_{i,j=n_{k-1}+1}^n a_{ij} X_i Y_j \longrightarrow 0 \quad \text{a. s.}$$

uniformly for $n \in (n_{k-1}, n_k]$.

Recalling (5.5), (5.6), (5.8), and (5.12) we have

$$\lim b_n^{-1} Q_n(X, Y) = 0 \quad \text{a. s.}$$

Lemma 5.3 is proved.

LEMMA 5.4. 1) $\lim 2^{-k} \sum_{kk} a_{ij}(X_i X_j - \mathbf{E} X_i X_j) = 0$ a. s. holds if and only if (5.3) is satisfied.

2) $\lim 2^{-k} \sum_{kk} a_{ij} X_i X_j = 0$ a. s. holds if and only if (5.4) is satisfied.

The proof is analogous to the proof of Theorem 4.4.

The assertion of Theorem 4.1 follows from Lemmas 5.1–5.4 and from Theorem 4.4.

Concluding remark. The proofs of Lemmas 5.1–5.3 are not dependent on the fact that X and Y are Gaussian sequences. Hence it is proved for arbitrary sequences (X_n) of symmetric r. v.'s that the SLLN $\lim b_n^{-1} Q_n(X, Y) = 0$ a. s. holds if and only if (5.3) and (5.4) are satisfied.

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